A number of different attempts have been made to describe the observed change in the rotational rate at the outer edge of a galaxy using different mechanisms. For instance attempts to fit data using modified Newtonian Dynamics (MOND) have been done.[1] Mechanisms involving bulk drag, shearing and galactic interaction with a background magnetic field have also been proposed. Astronomers have also suggested dark matter in the outer regions of galaxies as a source of additional gravitational effects.

The model outlined in this paper, although based on a number of dynamical simplifications, does not require any special assumptions. An outside observer viewing a galaxy from a great distance will be able to measure the frame dragging due to the rotation of the galaxy, by noting the slowing down of the rotational rate at the edge of the galaxy.

While we normally consider frame dragging when studying compact sources with high rotational rates, we should not overlook frame dragging in large sources with small rotational rates. Although the Lense-Thirring frame dragging is a minute effect, due to a galaxy's size and angular momentum this minute effect would be observable at galactic scale.

1 An Additional Effect due to the Rotation of an Attracting Body

In Newtonian mechanics an object in orbit around a non-rotating massive object such as a star has its orbital dynamics determined solely by the mass of the attracting body and the initial conditions of the orbiting object. In relativity the rotation of the attracting body will have an additional effect on the orbiting object.

In the naive view, the additional effect due to the rotation of the attracting body is a direct result of the established equivalence between mass and energy, namely

$$\Delta E = \Delta m c^2 \tag{1}$$

In this view a rotating disk has an energy due to motion of

$$\Delta E = \frac{1}{2} I \omega^2 \tag{2}$$

where ω is the rate of rotation and I is the moment of inertia. For a disk the moment of inertia is

$$I = \frac{1}{2} M R^2 \tag{3}$$

where M is the (rest) mass of the disk and R is the radius of the disk. This means that in the naive view

$$\Delta m = \frac{\Delta E}{c^2} = \frac{1}{c^2} \left(\frac{1}{2} M R^2\right) \frac{1}{2} \omega^2 \tag{4}$$

The energy of rotation would be perceived as an incremental change in the mass of the spinning homogenous disk given by

$$\Delta m = \frac{1}{4} \frac{M R^2 \omega^2}{c^2} \tag{5}$$

This incremental change in the mass of the spinning disk results in an incremental change in the gravitational acceleration of the attracting body, namely

$$\delta a = \frac{G \Delta m}{R^2} \approx \frac{G M \omega^2}{c^2} \tag{6}$$

where G is the Gravitational constant. Of course, galaxies are not homogeneous bodies and so

$$\Delta m = \beta \frac{M R^2 \omega^2}{c^2} \tag{7}$$

where β is some parameter specific to the distribution of mass in a particular galaxy.

For a body in orbit around a rotating mass we have a slightly different dynamics of motion compared to a body in orbit around a non-rotating attracting body with the same mass. To go beyond the naive view requires the study of the metric of a spherically symmetric rotating source.

2 The Lense-Thirring Metric and Frame Dragging

The geometry outside a static, spherically symmetric source can be described by the simple line element

$$ds^2 = -(1 - \frac{2GM}{c^2 r})(cdt)^2 + \frac{1}{(1 - \frac{2GM}{c^2 r})}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2) \qquad (8)$$

To accomodate rotation at small velocities, start with the Schwarzschild metric and add an additional cross term in dt $d\phi$, to arrive at the Lense-Thirring metric namely

$$\begin{aligned} ds^2 = -(1 - \frac{2GM}{c^2 r})(cdt)^2 + \frac{1}{(1 - \frac{2GM}{c^2 r})}dr^2 &+ r^2(d\theta^2 + sin^2\theta d\phi^2) \\ &- 2\frac{2GMa}{c^2 r}\sin^2\theta dt d\phi, \quad (9) \end{aligned}$$

where a is the angular momentum of the source divided by the mass of the source.

Unlike the Schwarzschild metric, the Lense-Thirring metric cannot be diagonalized by any coordinate transformation unless an explicit time dependence of the metric components is admitted. The non-diagonality of the Lense-Thirring metric is the cause of an effect known as "frame dragging".

Imagine that a small object is travelling in a Lense-Thirring Field at an instantaneously constant radial coordinate r (that is at that moment $dr/dt = 0$), but the second derivative of r with respect to time may be non-zero.

Consider the radial geodesic equation [2].

$$\frac{d}{ds}(g_{rr}\frac{dr}{ds}) = \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial r}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} \tag{10}$$

Since the radial value r would be considered as instantaneously constant, the second derivative with respect to s is given by

$$\frac{d^2r}{ds^2} = \frac{1}{2g_{rr}}\frac{\partial g_{\mu\nu}}{\partial r}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} \tag{11}$$

which expanded out becomes

$$\frac{d^2r}{ds^2} = \frac{1}{2g_{rr}}(\ \frac{\partial g_{tt}}{\partial r}(cdt)^2 + 2c\frac{\partial g_{t\phi}}{\partial r}\frac{dt}{ds}\frac{d\phi}{ds} + \frac{\partial g_{\phi\phi}}{\partial r}(\frac{d\phi}{ds})^2 + \frac{\partial g_{\theta\theta}}{\partial r}(\frac{d\theta}{ds})^2\ ) \tag{12}$$

By inspection it is possible to see that the part of the radial acceleration that is due to the source's rotation is given by

$$(\frac{d^2r}{ds^2})_{rot} = \frac{c}{g_{rr}}\frac{\partial g_{t\phi}}{\partial r}\frac{dt}{ds}\frac{d\phi}{ds} \tag{13}$$

By expanding the right hand side and retaining only those terms with the same order of magnitude as Gm/c^2r we see that

$$(\frac{d^2r}{ds^2})_{rot} = \frac{2GMa\sin^2\theta}{c^2r^2}\frac{dt}{ds}\frac{d\phi}{ds} \tag{14}$$

When the object moves slowly ds differs sufficiently little from cdt so that we can assume $ds \approx cdt$. Thus we see that (let $\theta = \frac{\pi}{2}$)

$$(\frac{d^2r}{dt^2})_{rot} = \frac{2GMa}{c^2R^2}\ \Omega_o \tag{15}$$

where $\frac{d\phi}{dt} = \Omega_o$. As you can see this is an additional acceleration due to the cross term in the Lense-Thirring metric. If the object moves in the same direction as the field rotates, the acceleration is directed outwards, otherwise, if the object moves counter to the source's rotation the acceleration is directed inwards.

If we let

$$a = \kappa\ \Omega_o R^2 \tag{16}$$

where as stated previously, a is the angular momentum of the source divided by the mass of the source, then

$$(\frac{d^2r}{dt^2})_{rot} = \kappa \ \frac{2GM\Omega_o^2}{c^2} \tag{17}$$

which has a form similar to the naive expression for the incremental change in the gravitational acceleration outlined in the first section of this paper.

3 The Change of Rotational Rate

Let us once again use a naive model. In a stable circular orbit in a Newtonian gravitational field we have

$$\frac{v^2}{R} = \frac{GM}{R^2} \tag{18}$$

If we substitute $v = \Omega_o R$ into this equation we arrive at the Kepler relationship

$$\Omega_o^2 = \frac{GM}{R^3} \tag{19}$$

Now imagine a small additional acceleration $\delta\ \alpha$ in the outward direction such that

$$\frac{v^2}{R} = \frac{GM}{R^2} + \delta\ \alpha \tag{20}$$

where $\delta\ \alpha$ is given by (on the equitorial plane $\theta = \pi/2$),

$$\delta\ \alpha = \frac{2GMa}{c^2R^2}\ \Omega \tag{21}$$

Then what we find is

$$\Omega^2 = \frac{GM}{R^3}\ (1 + \frac{2a}{c^2}\ \Omega) = \ \Omega_o^2\ (1 + \frac{2a}{c^2}\ \Omega) \tag{22}$$

Solving this quadratic equation we arrive at

$$\Omega \approx \Omega_o(\ 1 + \frac{a}{c^2} + \frac{a^2}{2c^4} \ ) \tag{23}$$

Keeping the first two terms we see then that

$$\Omega \approx \Omega_o \ (1 + \frac{a}{c^2}) \tag{24}$$

Again, if we assume that the angular momentum parameter a is given by

$$a \approx \kappa \ R^2 \Omega_o \tag{25}$$

then to simplest approximation the change of the rotational rate as a function of the radius R is given by

$$\Omega_{rot} \approx \Omega_0 \ (1 + \kappa \ \frac{R^2 \Omega_0}{c^2}) \tag{26}$$

Looking at a typical galaxy we can do a back of the envelope calculation which shows that at a kiloparsec radius from its centre the change in rotational rate is on the order of one part in 10^{10}.

As can be seen, although Lense-Thirring frame dragging is a minute effect, due to a galaxy's size and angular momentum this minute effect is observable at galactic scale. If the centre of galaxies include massive black holes then the Lense-Thirring dragging effect is expected to be more pronounced.

4 Discussion

The model outlined in this paper, although based on a number of dynamical simplifications, does not require special assumptions. While we normally consider frame dragging when studying compact sources with high rotational rates, we should not overlook frame dragging in large sources with small rotational rates.

We see then that a star which is in orbit around a rotating galaxy will orbit the centre at a different rotational rate than were it in orbit around a non-rotating galaxy. This may account for the peculiarities found in galctic astronomy.

5 Acknowledgements

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