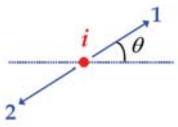
1.3

Particle decay



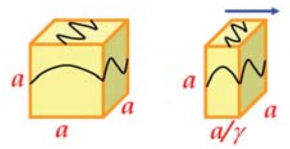
Final state in Born approximation (plane wave):

$$\Psi_1 = N \exp i(p \cdot x) = N \exp i(p^{\mu} x_{\mu})$$

Non - relativistic normalisation:

$$\int \Psi^* \Psi dV = N^2 V = 1 \Rightarrow N = \frac{1}{\sqrt{V}}$$

Non – relativistic final state density by counting states in cube V=a^3.



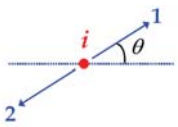
However, relativistically, V contracts by γ=E/m

To obtain L.I. normalize to 2E particles per volume:

$$\int \Psi'^* \Psi' dV = 2E \Rightarrow N = \frac{1}{\sqrt{2EV}}$$

1.3

Particle decay rate



$$\Gamma_{fi} = 2\pi \int |T_{fi}|^2 \rho(E_f)$$

T_{fi} – transition amplitude
Relativistic treatment of final state density:

$$\rho(E_f) = \left|\frac{dn}{dE}\right|_{E_f} = \int \frac{dn}{dE} \delta(E - E_i) dE$$

using $E_f = E_i$

Golden rule:

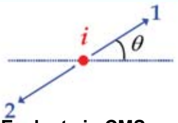
$$\Gamma_{fi} = 2\pi \int |T_{fi}|^2 \delta(E_i - E) dn$$
$$= 2\pi \int |T_{fi}|^2 \delta(E_i - E_1 - E_2) \frac{d^3\vec{p}_1}{(2\pi)^3}$$
$$= (2\pi)^4 \int |T_{fi}|^2 \delta(E_i - E_1 - E_2) \delta^3(\vec{p}_i - \vec{p}_1 - \vec{p}_2) \frac{d^3\vec{p}_1}{(2\pi)^3} \frac{d^3\vec{p}_2}{(2\pi)^3}$$
$$= \frac{(2\pi)^4}{2E_i} \int |M_{fi}|^2 \delta(E_i - E_1 - E_2) \delta^3(\vec{p}_i - \vec{p}_1 - \vec{p}_2) \frac{d^3\vec{p}_1}{(2\pi)^3 2E_1} \frac{d^3\vec{p}_2}{(2\pi)^3 2E_2}$$

L.I. matrix element M_{fi}: $M_{fi} = (2E_i \cdot 2E_1 \cdot 2E_2)^{\frac{1}{2}} T_{fi} = (2E_i \cdot 2E_1 \cdot 2E_2)^{\frac{1}{2}} \langle \Psi_1 \Psi_2 | \hat{H} | \Psi_i \rangle$

L.I. phase space (LIPS) for each final state particle: $\frac{d^3\vec{p}}{(2\pi)^3 2E}$

1.3

Decay rate of 2-body decay (I)



$$\Gamma_{fi} = \frac{(2\pi)^4}{2E_i}\int |M_{fi}|^2 \delta(E_i - E_1 - E_2)\delta^3(\vec{p}_i - \vec{p}_1 - \vec{p}_2)\frac{d^3\vec{p}_1}{(2\pi)^3 2E_1}\frac{d^3\vec{p}_2}{(2\pi)^3 2E_2}$$

Evaluate in CMS: $E_i = m_i,\ \vec{p}_i = 0$

Integrate over p_2

$$\Gamma_{fi} = \frac{1}{8\pi^2 m_i}\int |M_{fi}|^2 \delta(m_i - E_1 - E_2)\frac{d^3\vec{p}_1}{4E_1E_2}$$

with $d^3\vec{p}_1 = |\vec{p}_1|^2 d|\vec{p}_1| d\Omega$

$$\Gamma_{fi} = \frac{1}{32\pi^2 m_i}\int |M_{fi}|^2 \delta\left(m_i - \sqrt{m_1^2 + |\vec{p}_1|^2} - \sqrt{m_2^2 + |\vec{p}_1|^2}\right)\frac{|\vec{p}_1|^2 d|\vec{p}_1| d\Omega}{E_1E_2}$$

with

$$f(|\vec{p}_1|) = m_i - \sqrt{m_1^2 + |\vec{p}_1|^2} - \sqrt{m_2^2 + |\vec{p}_1|^2}$$

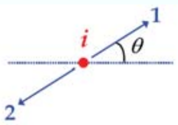
$$g(|\vec{p}_1|) = \frac{|\vec{p}_1|^2}{E_1E_2} = \frac{|\vec{p}_1|^2}{\sqrt{m_1^2 + |\vec{p}_1|^2}\sqrt{m_2^2 + |\vec{p}_1|^2}}$$

$$\Gamma_{fi} = \frac{1}{32\pi^2 m_i}\int |M_{fi}|^2 g(|\vec{p}_1|)\delta\left(f(|\vec{p}_1|)\right) d|\vec{p}_1| d\Omega$$

use property of δ – function: $\int g(p_1)\delta(f(p_1))dp_1 = \frac{1}{\left|\frac{df}{dp_1}\right|_{p^*}}\int g(p_1)\delta(p_1 - p^*)dp_1 = \frac{g(p^*)}{\left|\frac{df}{dp_1}\right|_{p^*}}$ with $f(p^*) = 0$

1.3

Decay rate of 2-body decay (II)



with

$$\frac{df(|\vec{p}_1|)}{d|\vec{p}_1|} = -\frac{|\vec{p}_1|}{\sqrt{m_1^2 + |\vec{p}_1|^2}} - \frac{|\vec{p}_1|}{\sqrt{m_2^2 + |\vec{p}_1|^2}} = -\frac{|\vec{p}_1|}{E_1} - \frac{|\vec{p}_1|}{E_2} = -|\vec{p}_1|\frac{E_1 + E_2}{E_1E_2}$$

$$\Gamma_{fi} = \frac{1}{32\pi^2 m_i}\int |M_{fi}|^2 \left.\frac{E_1E_2}{|\vec{p}_1|(E_1 + E_2)}\frac{|\vec{p}_1|^2}{E_1E_2}\right|_{p_1 = p^*} d\Omega$$

$$= \frac{|\vec{p}^*|}{32\pi^2 m_i^2}\int |M_{fi}|^2 d\Omega$$

Valid for all 2-body decays!

f(p*)=0:

$$f(p^*) = 0 = m_i - \sqrt{m_1^2 + p^{*2}} - \sqrt{m_2^2 + p^{*2}}$$

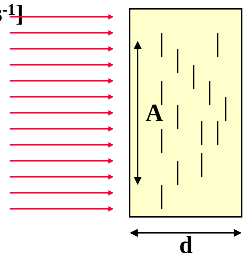
$$p^* = \frac{1}{2m_i}\sqrt{\left(m_i^2 - (m_1 + m_2)^2\right)\left(m_i^2 - (m_1 - m_2)^2\right)}$$

1.3

Cross section

Beam particle flux Φ [cm^{-2} s^{-1}]

$$\Phi = \frac{1}{A}\cdot\frac{\Delta N}{\Delta t} = \frac{N_{Beam}}{A}$$



Interacting area A' in area A

$$A' = N_{TAR}\cdot\sigma = \frac{\rho\cdot A\cdot d}{m_{mol}}\cdot N_A\cdot\sigma$$

with m_{mol} – molar mass
N_A - 6.02 10^{23}/mol
ρ - density

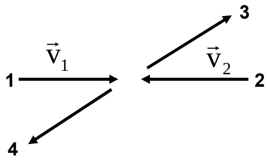
Cross section σ [cm^2]

$$\sigma = \frac{\text{number of reaction (of certain type) per unit time}}{\text{incoming flux}\cdot\text{number of scattering centers}} = \frac{N_{Reac}}{\frac{N_{Beam}}{A}\cdot N_{Tar}} = \frac{N_{Reac}}{N_{Beam}\cdot N_A\cdot\frac{\rho\cdot d}{m_{mol}}}$$

1.3

Cross section calculation

Consider scattering process 1 + 2 → 3 + 4



Reuse Fermi's Golden Rule:

$$\Gamma_{fi} = (2\pi)^4\int\left|T_{fi}\right|^2\delta(E_1+E_2-E_3-E_4)\delta^3(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)\frac{d^3\vec{p}_3}{(2\pi)^3}\frac{d^3\vec{p}_4}{(2\pi)^3}$$

T_{fi} is the transition matrix for a normalisation of 1 / unit volume.

Rate/volume = (flux of 1) x (number density of 2) x (cross section)

$$R = \Phi_1 n_2\sigma = n_1(\mathrm{v}_1+\mathrm{v}_2)n_2\sigma$$

Apply for 1 particle /unit volume

$$\Gamma_{fi} = (\mathrm{v}_1+\mathrm{v}_2)\sigma$$

$$\Downarrow$$

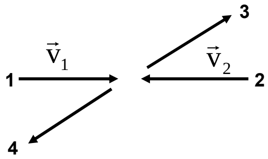
$$\sigma = \frac{(2\pi)^4}{(\mathrm{v}_1+\mathrm{v}_2)}\int\left|T_{fi}\right|^2\delta(E_1+E_2-E_3-E_4)\delta^3(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)\frac{d^3\vec{p}_3}{(2\pi)^3}\frac{d^3\vec{p}_4}{(2\pi)^3}$$

Not L.I. ! (use same steps as before to make it L.I.)

1.3

L.I. cross section calculation

Consider scattering process 1 + 2 → 3 + 4



$$\sigma = \frac{(2\pi)^4}{(v_1+v_2)} \int |T_{fi}|^2 \delta(E_1+E_2-E_3-E_4)\delta^3(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4) \frac{d^3\vec{p}_3}{(2\pi)^3}\frac{d^3\vec{p}_4}{(2\pi)^3}$$

$\Downarrow$ with $M_{fi} = (2E_1 \cdot 2E_2 \cdot 2E_3 \cdot 2E_4)^{\frac{1}{2}} T_{fi}$

$$\sigma = \frac{(2\pi)^{-2}}{2E_1 2E_2 (v_1+v_2)} \int |M_{fi}|^2 \delta(E_1+E_2-E_3-E_4)\delta^3(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4) \frac{d^3\vec{p}_3}{2E_3}\frac{d^3\vec{p}_4}{2E_4}$$

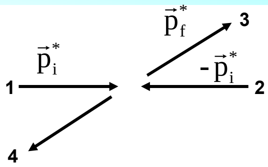
L.I. flux F

$$F = 2E_1 2E_2 (v_1+v_2) = 4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}$$

1.3

L.I. cross section 2→2 in CMS

Consider scattering process 1 + 2 → 3 + 4



$$\sigma = \frac{(2\pi)^{-2}}{2E_1 2E_2 (v_1+v_2)} \int |M_{fi}|^2 \delta(E_1+E_2-E_3-E_4)\delta^3(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4) \frac{d^3\vec{p}_3}{2E_3}\frac{d^3\vec{p}_4}{2E_4}$$

$$\sigma = \frac{(2\pi)^{-2}}{4|\vec{p}_i^*|\sqrt{s}} \int |M_{fi}|^2 \delta(\sqrt{s}-E_3-E_4)\delta^3(\vec{p}_3+\vec{p}_4) \frac{d^3\vec{p}_3}{2E_3}\frac{d^3\vec{p}_4}{2E_4}$$

$$= \frac{(2\pi)^{-2}}{4|\vec{p}_i^*|\sqrt{s}} \frac{|\vec{p}_f^*|}{4\sqrt{s}} \int |M_{fi}|^2 d\Omega^*$$

$$= \frac{1}{64\pi^2 s} \frac{|\vec{p}_f^*|}{|\vec{p}_i^*|} \int |M_{fi}|^2 d\Omega^*$$

1.3

Lorentz - invariant cross section

General Lorentz invariant form of n – particle phase space :

$$d\Phi_n(P; p_1,\dots, p_n) = \delta^4\left(P - \sum_{i=1}^{n} p_i\right)\prod_{i=1}^{n}\frac{d^3 p_i}{(2\pi)^3 2E_i}$$

Lips= `Lorentz invariant phase space´

PDG referemce:
http://www-pdg.lbl.gov/2011/reviews/rpp2011-rev-kinematics.pdf

Differential cross section for reaction 1 + 2 → 3 + … + (n+2)

$$d\sigma = \frac{(2\pi)^4 |M_{fi}|^2}{4\sqrt{(p_1\cdot p_2 - m_1^2 m_2^2)}}\, d\Phi_n(P = p_1 + p_2; p_3,\dots, p_{n+2})$$

Lorentz invariant flux factor

'fixed target': $p_1 = (E_1, \vec{p}_1)$, $p_2 = (m_2, 0)$ — **in rest frame of m_2:** $\sqrt{(p_1\cdot p_2)^2 - m_1^2 m_2^2} = m_2 p_{1,lab}$

'collider': $p_1 = (E_1, \vec{p})$, $p_2 = (E_2, -\vec{p})$, $\sqrt{s} = E_1 + E_2$ — **in CMS:** $\sqrt{(p_1\cdot p_2)^2 - m_1^2 m_2^2} = p_1^* \sqrt{s}$

2.

Description of particles

Reminder: Schrödinger equation $H\Psi = i\frac{\partial}{\partial t}\Psi, \quad H = T + V$

For free particle: $-\frac{\vec{\nabla}^2}{2m}\Psi = i\frac{\partial}{\partial t}\Psi$

Solution: $\Psi = N \exp i(p\cdot x) = N \exp i(\vec{p}\cdot\vec{x} - Et)$

Probability density: $\rho = \Psi^*\Psi = |\Psi|^2 = N^2$

Current: $\vec{j} = \frac{1}{2mi}(\Psi^*\vec{\nabla}\Psi - \Psi\vec{\nabla}\Psi^*) = |N|^2\frac{\vec{p}}{m} = |N|^2\vec{v}$

Continuity equation: $\vec{\nabla}\cdot\vec{j} + \frac{\partial\rho}{\partial t} = 0$

Task: replace classical E-p relation by relativistic expression

2.1

Klein – Gordon equation

For free particle:

$$E^2 = |\vec{p}|^2 + m^2, \quad \vec{p} = -i\vec{\nabla}, \quad E = i\partial/\partial t$$

$$\frac{\partial^2}{\partial t^2}\Psi = \left(\vec{\nabla}^2 - m^2\right)\Psi$$

$$\left(\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 + m^2\right)\Psi = \left(\partial^\mu \partial_\mu + m^2\right)\Psi = 0$$

Plane wave solution:

$$\Psi = N \exp i(p \cdot x) = N \exp i(\vec{p} \cdot \vec{x} - Et)$$

$$\Downarrow \text{insert}$$

$$-E^2\Psi = \left(-|\vec{p}|^2 - m^2\right)\Psi$$

$$E = \pm\sqrt{|\vec{p}|^2 + m^2}$$

Negative energy solution exists (?).

2.1

Probability density in KG equation

$$\frac{\partial^2}{\partial t^2}\Psi = \vec{\nabla}^2\Psi - m^2\Psi \qquad \text{(KG)}$$

$$i\Psi^*(KG) - i\Psi(KG)^* :$$

$$i\Psi^* \frac{\partial^2}{\partial t^2}\Psi - i\Psi \frac{\partial^2}{\partial t^2}\Psi^* = i\Psi^*\left(\vec{\nabla}^2\Psi - m^2\Psi\right) - i\Psi\left(\vec{\nabla}^2\Psi^* - m^2\Psi^*\right)$$

$$\frac{\partial}{\partial t}\underbrace{i\left(\Psi^* \frac{\partial}{\partial t}\Psi - \Psi \frac{\partial}{\partial t}\Psi^*\right)}_{\rho} = \vec{\nabla} \cdot \underbrace{i\left(\Psi^*\vec{\nabla}\Psi - \Psi\vec{\nabla}\Psi^*\right)}_{-\vec{j}}$$

Plane wave solution:

$$\Psi = N \exp i(p \cdot x) = N \exp i(\vec{p} \cdot \vec{x} - Et)$$

$$\Downarrow \text{insert}$$

$$\rho = i|N|^2(-iE - iE) = 2E|N|^2$$

$$\vec{j} = 2\vec{p}|N|^2$$

Negative probability for negative energy solution (?).

2.2

Dirac equation

Paul Dirac (1928) 1st order equation

$$H\Psi = (\vec{\alpha}\cdot\vec{p} + \beta m)\Psi = i\frac{\partial}{\partial t}\Psi, \qquad \vec{p} = -i\vec{\nabla}$$

Explicitely:

$$\left(i\tfrac{\partial}{\partial t}\right)\Psi = \left(-i\alpha_x\tfrac{\partial}{\partial_x} - i\alpha_y\tfrac{\partial}{\partial y} - i\alpha_z\tfrac{\partial}{\partial_z} + \beta m\right)\Psi$$

$$-\tfrac{\partial^2}{\partial t^2}\Psi = \left(-i\alpha_x\tfrac{\partial}{\partial_x} - i\alpha_y\tfrac{\partial}{\partial y} - i\alpha_z\tfrac{\partial}{\partial_z} + \beta m\right)\left(-i\alpha_x\tfrac{\partial}{\partial_x} - i\alpha_y\tfrac{\partial}{\partial y} - i\alpha_z\tfrac{\partial}{\partial_z} + \beta m\right)\Psi$$

KG – like ->

$$= \Big(-\alpha_x^2\tfrac{\partial^2}{\partial_x^2} - \alpha_y^2\tfrac{\partial^2}{\partial_y^2} - \alpha_z^2\tfrac{\partial^2}{\partial_z^2} + \beta^2 m^2$$
$$-(\alpha_x\alpha_y + \alpha_y\alpha_x)\tfrac{\partial^2}{\partial_x\partial_y} - (\alpha_y\alpha_z + \alpha_z\alpha_y)\tfrac{\partial^2}{\partial_y\partial_z} - (\alpha_z\alpha_x + \alpha_x\alpha_z)\tfrac{\partial^2}{\partial_z\partial_x}$$
$$-(\alpha_x\beta + \beta\alpha_x)m\tfrac{\partial}{\partial_x} - (\alpha_y\beta + \beta\alpha_y)m\tfrac{\partial}{\partial_y} - (\alpha_z\beta + \beta\alpha_z)m\tfrac{\partial}{\partial z} \Big)\Psi$$

Consistency with KG requires:

$$\alpha_x^2 = \alpha_y^2 = \alpha_z^2 = \beta^2 = 1$$
$$\alpha_j\alpha_k + \alpha_k\alpha_j = 0 \qquad (j \neq k)$$
$$\alpha_j\beta + \beta\alpha_j = 0$$

4 anticommuting 4x4 matrices needed!

2.2

Solution of Dirac equation

Four component Spinor $\Psi = \begin{pmatrix}\Psi_1\\ \Psi_2\\ \Psi_3\\ \Psi_4\end{pmatrix}$

Further insights requires choice of 4x4 matrices.

Here: Dirac – Pauli representation

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I\end{pmatrix}, \qquad \alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0\end{pmatrix}$$

$$\text{with}\ \ \mathrm{I} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\ \ \sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\ \ \sigma_y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\ \ \sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},$$

2.2

Dirac γ – matrices

$$\left(-i\alpha_x \tfrac{\partial}{\partial_x} - i\alpha_y \tfrac{\partial}{\partial y} - i\alpha_z \tfrac{\partial}{\partial_z} + \beta m\right)\psi = i\tfrac{\partial}{\partial t}\psi \,\Big|\cdot -\beta$$

$$\left(i\beta\alpha_x \tfrac{\partial}{\partial_x} + i\beta\alpha_y \tfrac{\partial}{\partial y} + i\beta\alpha_z \tfrac{\partial}{\partial_z} - \beta^2 m\right)\psi = -i\beta \tfrac{\partial}{\partial t}\psi$$

$$\Downarrow \gamma^\mu \equiv \left(\beta, \beta\alpha_\mu\right)$$

$$\left(i\gamma^0 \tfrac{\partial}{\partial t} + i\gamma^1 \tfrac{\partial}{\partial x} + i\gamma^2 \tfrac{\partial}{\partial y} + i\gamma^3 \tfrac{\partial}{\partial z} - \beta^2 m \right)\psi = 0$$

$$\Downarrow \partial_\mu = \left(\tfrac{\partial}{\partial t}, \tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial z}\right)$$ **covariant derivative**

$$\left(i\gamma^\mu \partial_\mu - m \right)\psi = 0$$ **covariant form of Dirac equation**

2.2

Properties of γ - matrices

$$\gamma^\mu \equiv \left(\beta, \beta\alpha_\mu\right)$$

$$\left(\gamma^0\right)^2 = 1$$

$$\left(\gamma^1\right)^2 = \left(\gamma^2\right)^2 = \left(\gamma^3\right)^2 = -1$$

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}, \qquad \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 0 \qquad (\mu \neq \nu)$$

$$\gamma^{0+} = \gamma^0 \quad \text{Hermitian}$$

$$\gamma^{j+} = -\gamma^j \quad \text{j} = 1,2,3 \ \text{anti-Hermitian}$$

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3, \quad \gamma^5\gamma^\mu + \gamma^\mu\gamma^5 = 0, \quad \gamma^{5+} = \gamma^5$$

Dagger symbol: $(\ )^+$