2.2

Pauli – Dirac representation of γ - matrices

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}$$

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \ \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \ \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \ \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\gamma^5 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Example:

$$\gamma^{2+} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}^+ = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} = -\gamma^2$$

transpose → **conjugate** →

2.2

Adjoint equation

Dirac eq.:

$$(\ i\gamma^\mu \partial_\mu - m\ )\psi = 0$$

$$(\ i\gamma^0 \tfrac{\partial}{\partial t} + i\gamma^k \tfrac{\partial}{\partial x_k} - m\ )\psi = 0$$

Hermitian conjugate

(Dirac eq.:)⁺

$$-i\tfrac{\partial}{\partial t}\psi^+\gamma^{0+} - i\tfrac{\partial}{\partial x_k}\psi^+\gamma^{k+} - m\psi^+ = 0$$

$$-i\tfrac{\partial}{\partial t}\psi^+\gamma^0 - i\tfrac{\partial}{\partial x_k}\psi^+(-\gamma)^k - m\psi^+ = 0 \quad \left|\cdot\gamma^0\right.$$

$$-i\tfrac{\partial}{\partial t}\psi^+\gamma^0\gamma^0 - i\tfrac{\partial}{\partial x_k}\psi^+(-\gamma^k\gamma^0) - m\psi^+\gamma^0 = 0 \quad \left|\gamma^k\gamma^0 = -\gamma^0\gamma^k\right.$$

$$i\tfrac{\partial}{\partial t}\psi^+\gamma^0\gamma^0 + i\tfrac{\partial}{\partial x_k}\psi^+\gamma^0\gamma^k + m\psi^+\gamma^0 = 0 \quad \left|\overline{\psi} \equiv \psi^+\gamma^0\right.$$

$$i\partial_\mu\overline{\psi}\gamma^\mu + m\overline{\psi} = 0$$

Adjoint row spinor $\overline{\psi}$

2.2

Adjoint spinor

$$\overline{\psi} = \psi^+ \gamma^0 = (\psi^*)^T \gamma^0$$
$$= (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$
$$= (\psi_1^*, \psi_2^*, -\psi_3^*, -\psi_4^*)$$

2.2

Probability density and currents

$$\overline{\psi} \cdot \Big| \quad (i\gamma^\mu \partial_\mu - m) \psi = 0$$
$$+$$
$$i\partial_\mu \overline{\psi} \gamma^\mu + m \overline{\psi} = 0 \quad \Big| \cdot \psi$$
$$\Downarrow$$
$$i\overline{\psi}\gamma^\mu \partial_\mu \psi - m\overline{\psi}\psi + i(\partial_\mu \overline{\psi})\gamma^\mu \psi + m\overline{\psi}\psi = 0$$
$$i\overline{\psi}\gamma^\mu \partial_\mu \psi + i(\partial_\mu \overline{\psi})\gamma^\mu \psi = 0$$
$$\partial_\mu (\overline{\psi}\gamma^\mu \psi) = 0$$

current $j^\mu = \overline{\psi}\gamma^\mu \psi = \begin{pmatrix} \rho \\ \vec{j} \end{pmatrix}$

continuity eq. $\partial_\mu j^\mu = 0$

probability $\rho = \overline{\psi}\gamma^0 \psi = \psi^+ \psi = \sum_{i=1}^{4} |\psi_i|^2 > 0$

Electron current:
$j^\mu = (-e)\overline{\psi}\gamma^\mu \psi$

For comparison:
Boson current:
(from KG – equation)
$j^\mu = (-e) 2 p^\mu$

2.3

Free particle spinors

Easiest case: **free particle at rest:** $(i\gamma^{\mu}\partial_{\mu} - m)\psi = 0$

$$\psi = u(E,\vec{p})e^{i(\vec{p}\cdot\vec{r}-Et)}$$

$$(\gamma^0 E - \gamma^1 p_x - \gamma^2 p_y - \gamma^3 p_z - m)u = 0$$

$$(\gamma^{\mu} p_{\mu} - m)u = 0$$

$$\Downarrow \ \vec{p} = 0, \psi = u(E,\vec{p})e^{-iEt}$$

$$E\gamma^0 u - mu = 0$$

$$E\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\\ \varphi_4\end{pmatrix} = m\begin{pmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\\ \varphi_4\end{pmatrix}$$

4 independent solutions:

$$\underbrace{\psi_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}e^{-imt}, \psi_2 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}e^{-imt}}_{E=m>0}, \underbrace{\psi_3 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}e^{+imt}, \psi_4 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}e^{+imt}}_{E=-m<0}$$

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Derivation of plane wave solution

Find general solution:

$$\psi = u(E,\vec{p})e^{i(\vec{p}\cdot\vec{r}-Et)}$$

$$(\gamma^{\mu} p_{\mu} - m)u = (\gamma^0 E - \gamma^1 p_x - \gamma^2 p_y - \gamma^3 p_z - m)u$$

$$= \begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix}E - \begin{pmatrix}0 & \vec{\sigma}\\ -\vec{\sigma} & 0\end{pmatrix}\cdot\vec{p} - \begin{pmatrix}I & 0\\ 0 & I\end{pmatrix}m$$

$$= \begin{pmatrix}(E-m)I & -\vec{\sigma}\cdot\vec{p}\\ \vec{\sigma}\cdot\vec{p} & -(E+m)I\end{pmatrix}$$

Ansatz taking into account substructure of 4x4 matrix with $u = \begin{pmatrix}u_A\\ u_B\end{pmatrix}$

$$\begin{pmatrix}(E-m)I & -\vec{\sigma}\cdot\vec{p}\\ \vec{\sigma}\cdot\vec{p} & -(E+m)I\end{pmatrix}\begin{pmatrix}u_A\\ u_B\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

Coupled equations:

$$(\vec{\sigma}\cdot\vec{p})u_B = (E-m)u_A$$
$$(\vec{\sigma}\cdot\vec{p})u_A = (E+m)u_B$$

$$\vec{\sigma}\cdot\vec{p} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}p_x + \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}p_y + \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}p_z$$

$$= \begin{pmatrix}p_z & p_x - ip_y\\ p_x + ip_y & -p_z\end{pmatrix}$$

$$u_B = \frac{\vec{\sigma}\cdot\vec{p}}{E+m}u_A = \frac{1}{E+m}\begin{pmatrix}p_z & p_x - ip_y\\ p_x + ip_y & -p_z\end{pmatrix}u_A$$

Explicite solutions by making an arbitrary choice for u_A and u_B (motivated by particle-at-rest solutions):

$$u_A^{(1)} = \begin{pmatrix}1\\0\end{pmatrix}, u_A^{(2)} = \begin{pmatrix}0\\1\end{pmatrix}, u_B^{(3)} = \begin{pmatrix}1\\0\end{pmatrix}, u_B^{(4)} = \begin{pmatrix}0\\1\end{pmatrix}$$

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Plane wave solutions

$$u_1 = N_1 \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix}, \quad u_2 = N_2 \begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}, \quad u_3 = N_3 \begin{pmatrix} \frac{p_z}{E-m} \\ \frac{p_x+ip_y}{E-m} \\ 1 \\ 0 \end{pmatrix}, \quad u_4 = N_4 \begin{pmatrix} \frac{p_x-ip_y}{E-m} \\ \frac{-p_z}{E-m} \\ 0 \\ 1 \end{pmatrix},$$

All 4 solutions give $E^2 = |\vec{p}|^2 + m^2$

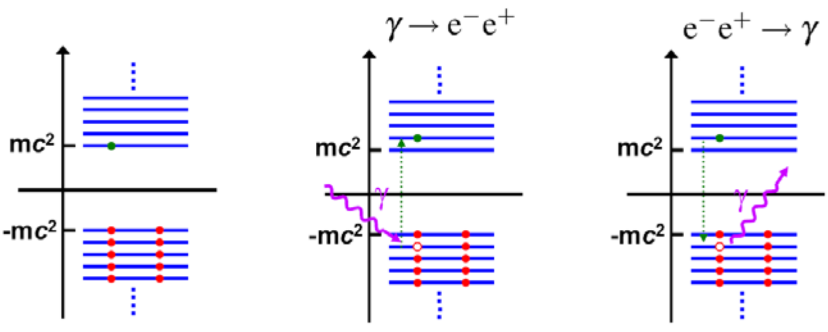
Comparison to vanishing momentum solutions gives rise to the statement:

u_1,u_2 correspond to E>0 particle at rest solutions
u_3,u_4 correspond to E<0 particle at rest solutions

2.3

Interpretation of negative energy solutions

Dirac picture:

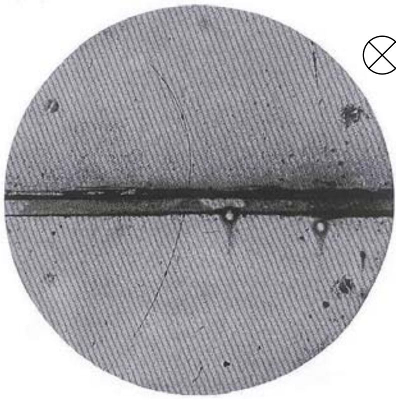


All lower states are filled.
Electrons obey Fermi – Dirac – statistics (Pauli exclusion principle).

2.3

Discovery of positron

Cloud chamber picture, C.D. Anderson (1933)



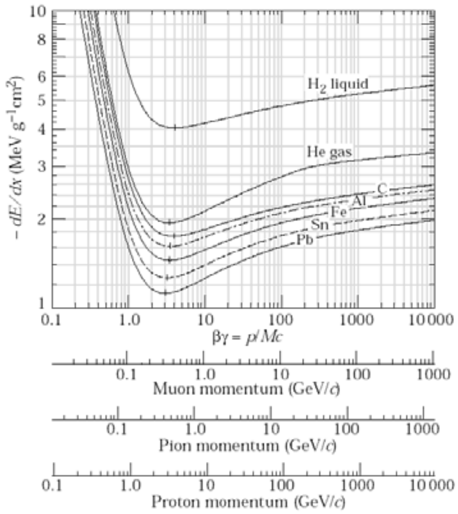
$\otimes \vec{B}$

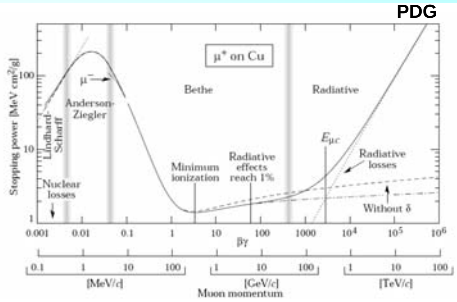
6mm Pb plate

2.3

Energy loss of charged particles

PDG





units: $-\frac{dE}{d\varepsilon} = -\frac{1}{\rho}\cdot\frac{dE}{dx} = z^2\frac{Z}{A}\cdot f(\beta, I)$

$$\frac{\text{MeV}}{\frac{\text{g}}{\text{cm}^2}} = \frac{1}{\frac{\text{g}}{\text{cm}^3}}\frac{\text{MeV}}{\text{cm}}$$

"dE/dx" normally given in MeV g^{-1} cm^2

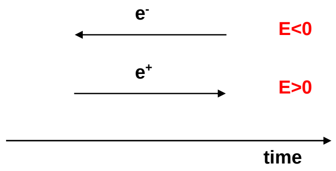
dE/dx independent of the mass of the projectile

BB valid only for 'heavy' particles ($m>m_\mu$)

2.3

Feynman – Stückelberg – Interpretation

Interpret a negative energy solution as a negative energy particle that propagates backward in time or equivalently positive energy antiparticle that propagates forward in time.



Note: in Feynman diagrams one keeps the negative energy direction of the arrow.

2.3

Anti-particle spinors

$$u_3 = N_3 \begin{pmatrix} \frac{p_z}{E-m} \\ \frac{p_x + ip_y}{E-m} \\ 1 \\ 0 \end{pmatrix}, \quad u_4 = N_4 \begin{pmatrix} \frac{p_x - ip_y}{E-m} \\ \frac{-p_z}{E-m} \\ 0 \\ 1 \end{pmatrix}$$

E<0, particle

$$\downarrow \text{flip sign of E and } \vec{p}$$

$$v_1(E,\vec{p})e^{-i(\vec{p}\cdot\vec{r}-Et)} = u_4(-E,-\vec{p})e^{i(\vec{p}\cdot\vec{r}-Et)}$$
$$v_2(E,\vec{p})e^{-i(\vec{p}\cdot\vec{r}-Et)} = u_3(-E,-\vec{p})e^{i(\vec{p}\cdot\vec{r}-Et)}$$

E>0, antiparticle

$$\downarrow$$

$$v_1 = N_1' \begin{pmatrix} \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix}, \quad v_2 = N_2' \begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix},$$

2.3

Particle and anti-particle spinors

Alternative forms of solutions:

1) $$\psi_i = u_i(E,\vec{p})e^{i(\vec{p}\cdot\vec{r}-Et)}$$

$$\underbrace{u_1 = N_1\begin{pmatrix}1\\0\\\frac{p_z}{E+m}\\\frac{p_x+ip_y}{E+m}\end{pmatrix},\quad u_2 = N_2\begin{pmatrix}0\\1\\\frac{p_x-ip_y}{E+m}\\\frac{-p_z}{E+m}\end{pmatrix}}_{E=+\left|\sqrt{|\vec{p}|^2+m^2}\right|},\quad \underbrace{u_3 = N_3\begin{pmatrix}\frac{p_z}{E-m}\\\frac{p_x+ip_y}{E-m}\\1\\0\end{pmatrix},\quad u_4 = N_4\begin{pmatrix}\frac{p_x-ip_y}{E-m}\\\frac{-p_z}{E-m}\\0\\1\end{pmatrix}}_{E=-\left|\sqrt{|\vec{p}|^2+m^2}\right|},$$

2) $$\psi_i = \mathrm{v}_i(E,\vec{p})e^{-i(\vec{p}\cdot\vec{r}-Et)}$$

$$\underbrace{\mathrm{v}_1 = N\begin{pmatrix}\frac{p_x-ip_y}{E+m}\\\frac{-p_z}{E+m}\\0\\1\end{pmatrix},\quad \mathrm{v}_2 = N\begin{pmatrix}\frac{p_z}{E+m}\\\frac{p_x+ip_y}{E+m}\\1\\0\end{pmatrix}}_{E=+\left|\sqrt{|\vec{p}|^2+m^2}\right|},\quad \underbrace{\mathrm{v}_3 = N_1\begin{pmatrix}1\\0\\\frac{p_z}{E-m}\\\frac{p_x+ip_y}{E-m}\end{pmatrix},\quad \mathrm{v}_4 = N_2\begin{pmatrix}0\\1\\\frac{p_x-ip_y}{E-m}\\\frac{-p_z}{E-m}\end{pmatrix}}_{E=-\left|\sqrt{|\vec{p}|^2+m^2}\right|},$$

For calculations different spinor systems possible giving the same results:

$$\{u_1,u_2,u_3,u_4\},\quad \{\mathrm{v}_1,\mathrm{v}_2,\mathrm{v}_3,\mathrm{v}_4\},\quad \{u_1,u_2,\mathrm{v}_1,\mathrm{v}_2\}$$

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Wavefunction normalisation

Wavefunctions have to be normalized to 2E:

$$\int \rho dV = \int \psi^+\psi dV = \int \left(\psi^*\right)^T \psi dV \overset{!}{=} 2E$$

Example: $\psi = u_1(E,\vec{p})e^{i(\vec{p}\cdot\vec{r}-Et)}$

$$u_1 = N\begin{pmatrix}1\\0\\\frac{p_z}{E+m}\\\frac{p_x+ip_y}{E+m}\end{pmatrix}$$

$$\rho = u_1^+u_1 = |N|^2\begin{pmatrix}1 & 0 & \frac{p_z}{E+m} & \frac{p_x-ip_y}{E+m}\end{pmatrix}\begin{pmatrix}1\\0\\\frac{p_z}{E+m}\\\frac{p_x+ip_y}{E+m}\end{pmatrix}$$

$$= |N|^2\left(1+\frac{p_z^2}{(E+m)^2}+\frac{p_x^2+p_y^2}{(E+m)^2}\right)$$

$$= |N|^2\left(\frac{(E+m)^2+|\vec{p}|^2}{(E+m)^2}\right) = |N|^2\left(\frac{(E+m)^2+E^2-m^2}{(E+m)^2}\right)$$

$$= |N|^2\frac{2E}{E+m}$$

$$\Downarrow$$

$$N = \sqrt{E+m}$$

Same result for N with u_1, u_2, v_1, v_2

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Summary of solutions

Particle solutions: $\psi = u(E,\vec{p})e^{i(\vec{p}\cdot\vec{r}-Et)}$ satisfy $(\gamma^\mu p_\mu - m)u = 0$

$$u_1 = \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix}, \quad u_2 = \sqrt{E+m}\begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}$$

Anti-particle solutions: $\psi = \mathrm{v}(E,\vec{p})e^{-i(\vec{p}\cdot\vec{r}-Et)}$ satisfy $(\gamma^\mu p_\mu + m)u = 0$

$$\mathrm{v}_1 = \sqrt{E+m}\begin{pmatrix} \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix}, \quad \mathrm{v}_2 = \sqrt{E+m}\begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix}$$

These solutions have positive energies: $E = +\sqrt{|\vec{p}|^2 + m^2}$

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2.3

Spin

Spinors u_1, u_2, v_1, v_2 are not Eigenstates of S_z

$$\hat{S}_z = \tfrac{1}{2}\Sigma_z = \tfrac{1}{2}\begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

However, particles and antiparticles moving in z-direction are Eigenstates of S_z $\quad p_z = \pm|\vec{p}|$

$$u_1 = \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ \frac{\pm|\vec{p}|}{E+m} \\ 0 \end{pmatrix}, \quad u_2 = \sqrt{E+m}\begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{\mp|\vec{p}|}{E+m} \end{pmatrix}, \quad \mathrm{v}_1 = \sqrt{E+m}\begin{pmatrix} 0 \\ \frac{\mp|\vec{p}|}{E+m} \\ 0 \\ 1 \end{pmatrix}, \quad \mathrm{v}_2 = \sqrt{E+m}\begin{pmatrix} \frac{\pm|\vec{p}|}{E+m} \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\hat{S}_z u_1 = \tfrac{1}{2}u_1 \qquad \hat{S}_z^{(\mathrm{v})}\mathrm{v}_1 = -\hat{S}_z\mathrm{v}_1 = -\tfrac{1}{2}\mathrm{v}_1$$

$$\hat{S}_z u_2 = -\tfrac{1}{2}u_2 \qquad \hat{S}_z^{(\mathrm{v})}\mathrm{v}_2 = -\hat{S}_z\mathrm{v}_2 = -\tfrac{1}{2}\mathrm{v}_2$$

Note: the Eigenvalues are independent of the sign of p_z

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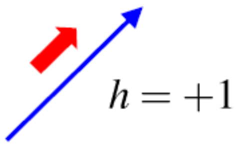
Helicity

The component of the spin along the particle direction of flight is a good quantum number:

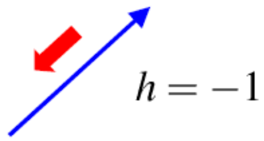
$$\left[\hat{H}, \hat{\sigma} \cdot \hat{p}\right] = 0$$

Therefore: definition of Helicity:

$$h \equiv \frac{\vec{S} \cdot \vec{p}}{|\vec{S}||\vec{p}|} = \frac{2\vec{S} \cdot \vec{p}}{|\vec{p}|} = \frac{2\vec{\Sigma} \cdot \vec{p}}{|\vec{p}|}$$



right handed



left handed

2.3

Helicity - Eigenstates

Find spinors with $\hat{p} = \frac{\vec{p}}{|\vec{p}|}$ **unit 3-momentum vector**

$$(\vec{\Sigma} \cdot \hat{p})u_\uparrow = +u_\uparrow$$

$$(\vec{\Sigma} \cdot \hat{p})u_\downarrow = -u_\downarrow$$

Eigenvalue equation:

$$\begin{pmatrix} \vec{\sigma} \cdot \vec{p} & 0 \\ 0 & \vec{\sigma} \cdot \vec{p} \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix} = \pm \begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

$$\Rightarrow \begin{cases} (\vec{\sigma} \cdot \hat{p})u_A = \pm u_A \\ (\vec{\sigma} \cdot \hat{p})u_B = \pm u_B \end{cases}$$

Coupled equations

Calculate explicitely for a particle moving into (θ,φ) direction:

$$\hat{p} = (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta)$$

$$\vec{\sigma} \cdot \hat{p} = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix} = \begin{pmatrix} \cos\vartheta & \sin\vartheta\cos\varphi - i\sin\vartheta\sin\varphi \\ \sin\vartheta\cos\varphi + i\sin\vartheta\sin\varphi & -\cos\vartheta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\vartheta & \sin\vartheta e^{-i\varphi} \\ \sin\vartheta e^{+i\varphi} & -\cos\vartheta \end{pmatrix}$$

$$u_A = \begin{pmatrix} a \\ b \end{pmatrix} = u_B : \quad \begin{pmatrix} \cos\vartheta & \sin\vartheta e^{-i\varphi} \\ \sin\vartheta e^{+i\varphi} & -\cos\vartheta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \pm \begin{pmatrix} a \\ b \end{pmatrix}$$

for helicity h = +/- 1

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Helicity – Eigenstates (II)

$$u_A = \begin{pmatrix} a \\ b \end{pmatrix} = u_B : \quad \begin{pmatrix} \cos\vartheta & \sin\vartheta e^{-i\varphi} \\ \sin\vartheta e^{+i\varphi} & -\cos\vartheta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \pm \begin{pmatrix} a \\ b \end{pmatrix}$$

for helicity h = +/- 1

$$\begin{matrix} \pm a\cos\vartheta \pm b\sin\vartheta e^{-i\varphi} = \pm a \\ \pm a\sin\vartheta e^{+i\varphi} \mp b\cos\vartheta = \pm b \end{matrix} \Rightarrow \pm a(1 - \pm\cos\vartheta) = \pm b\sin\vartheta e^{-i\varphi} \Rightarrow \frac{b}{a} = \frac{\pm 1 - \cos\vartheta}{\sin\vartheta} e^{i\varphi}$$

For right handed helicity state (h=1):

$$\frac{b}{a} = \frac{1-\cos\vartheta}{\sin\vartheta} e^{i\varphi} = \frac{2\sin^2\left(\frac{\vartheta}{2}\right)}{2\sin\frac{\vartheta}{2}\cos\frac{\vartheta}{2}} e^{i\varphi} = e^{i\varphi}\frac{\sin\frac{\vartheta}{2}}{\cos\frac{\vartheta}{2}}$$

$$\Downarrow$$

$$u_{A\uparrow} \propto \begin{pmatrix} \cos\frac{\vartheta}{2} \\ e^{i\varphi}\sin\frac{\vartheta}{2} \end{pmatrix}$$

$$\Downarrow$$

$$u_\uparrow = \begin{pmatrix} u_A \\ u_B \end{pmatrix} = \begin{pmatrix} N_A\cos\frac{\vartheta}{2} \\ N_A e^{i\varphi}\sin\frac{\vartheta}{2} \\ N_B\cos\frac{\vartheta}{2} \\ N_B e^{i\varphi}\sin\frac{\vartheta}{2} \end{pmatrix}$$

Page 38:

$$(\vec{\sigma}\cdot\vec{p})u_A = (E+m)u_B$$

$$\downarrow h=1$$

$$u_B = \frac{|\vec{p}|}{E+m} u_A$$

$$\downarrow$$

$$u_\uparrow = N \begin{pmatrix} \cos\frac{\vartheta}{2} \\ e^{i\varphi}\sin\frac{\vartheta}{2} \\ \frac{|\vec{p}|}{E+m}\cos\frac{\vartheta}{2} \\ \frac{|\vec{p}|}{E+m} e^{i\varphi}\sin\frac{\vartheta}{2} \end{pmatrix}$$

Same for h=-1.

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Helicity – Eigenstates – Summary

Particles

$$u_\uparrow = N \begin{pmatrix} \cos\frac{\vartheta}{2} \\ e^{i\varphi}\sin\frac{\vartheta}{2} \\ \frac{|\vec{p}|}{E+m}\cos\frac{\vartheta}{2} \\ \frac{|\vec{p}|}{E+m} e^{i\varphi}\sin\frac{\vartheta}{2} \end{pmatrix} \quad u_\downarrow = N \begin{pmatrix} -\sin\frac{\vartheta}{2} \\ e^{i\varphi}\cos\frac{\vartheta}{2} \\ \frac{|\vec{p}|}{E+m}\sin\frac{\vartheta}{2} \\ \frac{-|\vec{p}|}{E+m} e^{i\varphi}\cos\frac{\vartheta}{2} \end{pmatrix}$$

Anti-Particles

$$\mathrm{v}_\uparrow = N \begin{pmatrix} \frac{|\vec{p}|}{E+m}\sin\frac{\vartheta}{2} \\ -\frac{|\vec{p}|}{E+m} e^{i\varphi}\cos\frac{\vartheta}{2} \\ -\sin\frac{\vartheta}{2} \\ e^{i\varphi}\cos\frac{\vartheta}{2} \end{pmatrix} \quad \mathrm{v}_\downarrow = N \begin{pmatrix} \frac{|\vec{p}|}{E+m}\cos\frac{\vartheta}{2} \\ \frac{|\vec{p}|}{E+m} e^{i\varphi}\sin\frac{\vartheta}{2} \\ \cos\frac{\vartheta}{2} \\ e^{i\varphi}\sin\frac{\vartheta}{2} \end{pmatrix}$$

Normalisation: $N = \sqrt{E+m}$