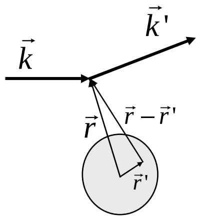
3.5

Formfactor

Scattering in the static potential of extended charge distribution:

$$V(r) = \frac{e}{4\pi}\int d^3\vec r\,' \frac{\rho(\vec r\,')}{|\vec r - \vec r\,'|} \qquad \text{with} \qquad \int d^3 r \rho(\vec r) = Z$$



First order perturbation theory:

$$M_{fi} = <\psi_f|V(\vec r)|\psi_i> = \int d^3\vec r e^{-i\vec k'\cdot\vec r} V(\vec r) e^{i\vec k\cdot\vec r}$$

$$\overset{\vec q = \vec k - \vec k'}{=} \iint d^3\vec r d^3\vec r\,' e^{i\vec q\cdot\vec r} \frac{e}{4\pi}\frac{\rho(\vec r\,')}{|\vec r - \vec r\,'|}$$

$$= \iint d^3\vec r d^3\vec r\,' e^{i\vec q\cdot(\vec r - \vec r\,')} e^{i\vec q\cdot\vec r\,'} \frac{e}{4\pi}\frac{\rho(\vec r\,')}{|\vec r - \vec r\,'|}$$

$$\overset{\vec R = \vec r - \vec r\,'}{=} \int d^3\vec R \frac{e}{4\pi|\vec R|} e^{i\vec q\cdot\vec R} \int d^3\vec r\,' e^{i\vec q\cdot\vec r\,'} \rho(\vec r\,')$$

$$= M_{fi}^{\text{point}} \cdot F(\vec q)$$

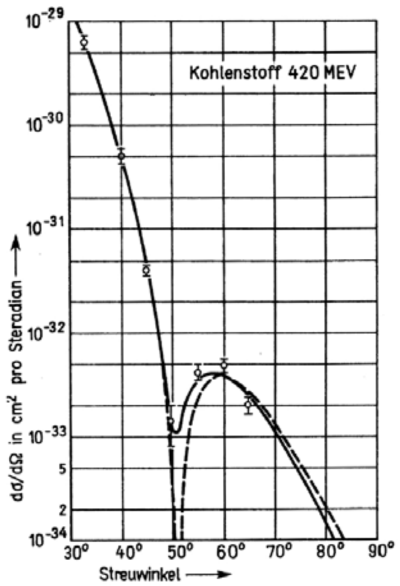
Factorisation: point source x formfactor

$$F(\vec q) = \frac{1}{Z}\int d^3 r \rho(\vec r) e^{i\vec q\cdot\vec r}$$

Fouriertransform of charge distribution

3.5

Elastic electron-nucleus scattering



R.Hofstadter,
Ann.Rev.Nucl.Sci. 7, 231 (1957)

Method for radius measurement:

For uniform sphere: 1. minimum at |q|R=4.5

Needs model of the charge distribution.

3.5

Properties of form factors

Normalization : $F(\vec{q}=0)=\frac{1}{Z}\int d^3r\rho(\vec{r})=1$

Spherical charge distribution $\rho(\vec{r})=Zf(r)$:

$$F(\vec{q})=\int d^3r e^{i\vec{q}\cdot\vec{r}}f(r)=2\pi\int_0^\infty drr^2 f(r)\int_{-1}^{1}d(\cos\Theta)e^{iqr\cos\Theta}$$

$$\int_{-1}^{1}d(\cos\Theta)e^{iqr\cos\Theta}=\frac{1}{iqr}e^{iqr\cos\Theta}\Big|_{-1}^{+1}=\frac{1}{iqr}\left(e^{iqr}-e^{-iqr}\right)=\frac{2}{qr}\sin qr$$

$$F\left(q^2\right)=\frac{4\pi}{q}\int drr\sin(qr)f(r)$$

Expansion (qR << 1):

$$F\left(q^2\right)=\frac{4\pi}{q}\int drr\left[qr-\tfrac{1}{6}q^3r^3+\ldots\right]f(r)$$

$$=4\pi\int_0^\infty drr^2 f(r)-\frac{4\pi q^2}{6}\int_0^\infty drr^4 f(r)$$

$$=\quad 1 \qquad - \quad \frac{q^2}{6}\left\langle r^2\right\rangle$$

$$\Rightarrow\left\langle r^2\right\rangle=-6\frac{dF(q^2)}{dq^2}\Bigg|_{q^2=0}$$

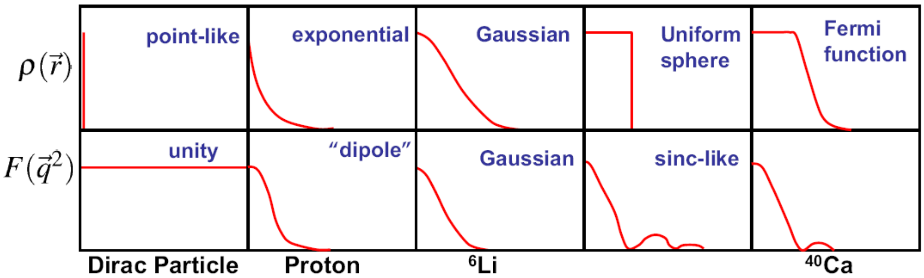
RMS radius is given by the slope of the formfactor at vanishing momentum transfer

3.5

Scattering from extended charge distributions

$$\left(\frac{d\sigma}{d\Omega}\right)_{Mott}=\left(\frac{d\sigma}{d\Omega}\right)_{Rutherford}\cdot\cos^2\frac{\theta}{2}\cdot\left|F\left(\vec{q}^{\,2}\right)\right|^2$$

Formfactor is an interference phenomenon, coherent scattering from all parts of the source.



Formfactor always reduce the corresponding point charge cross section.

3.3

Invariant Matrix element

$$L_e^{\mu\nu} = 2\left(k'^{\mu}k^{\nu} + k'^{\nu}k^{\mu} - (k'\cdot k - m^2)g^{\mu\nu}\right)$$ **Electron tensor**

$$L_{\mu\nu}^{muon} = 2\left(p'_{\mu}p_{\nu} + p'_{\nu}p_{\mu} - (p'\cdot p - M^2)g_{\mu\nu}\right)$$ **Muon tensor**

$$\overline{|M|^2} = \frac{e^2}{q^4} L_e^{\mu\nu}\cdot L_{muon,\mu\nu}$$

$$= 8\frac{e^2}{q^4}\left[(k'\cdot p')(k\cdot p) + (k'\cdot p)(k\cdot p') - m^2 p'\cdot p - M^2 k'\cdot k + 2m^2M^2\right]$$ **correct first order result!**

$$\downarrow m = 0, M = 0$$

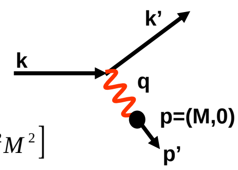
$$\overline{|M|^2} = \frac{8e^4}{(k-k')^4}\left[(k'\cdot p')(k\cdot p) + (k'\cdot p)(k\cdot p')\right]$$ **Spin averaged invariant transition matrix**

With Mandelstam variables:

$$s = (k+p)^2 = m^2 + M^2 + 2k\cdot p \rightarrow 2k\cdot p \approx 2k'\cdot p'$$
$$t = (k-k')^2 = m^2 + m^2 - 2k\cdot k' \rightarrow -2k\cdot k' \approx -2p\cdot p'$$
$$u = (k-p')^2 = m^2 + M^2 - 2k\cdot p' \rightarrow -2k\cdot p' \approx -2k'\cdot p$$
$$\Downarrow$$
$$\overline{|M|^2} = 2e^4\frac{s^2+u^2}{t^2}$$

3.5

Pointlike e$^-$-proton scattering



Use matrix element from e$^-\mu^-$ - scattering

$$\overline{|M|^2} = 8\frac{e^2}{q^4}\left[(k'\cdot p')(k\cdot p) + (k'\cdot p)(k\cdot p') - m^2 p'\cdot p - M^2 k'\cdot k + 2m^2M^2\right]$$

$$\downarrow m = 0$$

$$\overline{|M|^2} = \frac{8e^4}{(k-k')^4}\left[(k'\cdot p')(k\cdot p) + (k'\cdot p)(k\cdot p') - M^2 k'\cdot k\right]$$

$$\downarrow q = k - k', q^2 \approx -2k\cdot k', p' = k - k' + p, k^2 = k'^2 = 0$$

$$= \frac{8e^4}{q^4}\left[k'\cdot(k - k' + p)(k\cdot p) + (k'\cdot p)k\cdot(k - k' + p) + \tfrac{1}{2}M^2q^2\right]$$

$$= \frac{8e^4}{q^4}\left[\left(-\tfrac{1}{2}q^2 + k'\cdot p\right)k\cdot p + k'\cdot p\left(\tfrac{1}{2}q^2 + k\cdot p\right) + \tfrac{1}{2}M^2q^2\right]$$

$$= \frac{8e^4}{q^4}\left[-\tfrac{1}{2}q^2(k\cdot p - k'\cdot p) + 2(k'\cdot p)(k\cdot p) + \tfrac{1}{2}M^2q^2\right]$$

3.5

Pointlike e⁻-proton scattering

Evaluate in lab frame:

$p = (M,0,0,0)$

$$\overline{|M|^2} = \frac{8e^4}{q^4}\left[-\tfrac{1}{2}q^2(k\cdot p - k'\cdot p) + 2(k'\cdot p)(k\cdot p) + \tfrac{1}{2}M^2q^2\right]$$

$$= \frac{8e^4}{q^4}\left[-\tfrac{1}{2}q^2 M(E-E') + 2E'EM^2 + \tfrac{1}{2}M^2q^2\right]$$

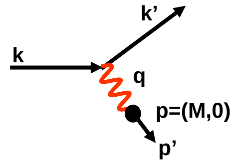
$$= \frac{8e^4}{q^4}2M^2E'E\left\{1 + \frac{q^2}{4E'E} - \frac{q^2}{2M^2}\frac{M(E-E')}{2E'E}\right\}$$

$$\downarrow q^2 \approx -2k\cdot k' \approx 2EE'(1-\cos\theta) = -4EE'\sin^2\tfrac{\theta}{2}$$

$$\downarrow \nu = (E-E') = -\frac{q^2}{2M}$$

$$= \frac{8e^4}{q^4}2M^2E'E\left\{\cos^2\tfrac{\theta}{2} - \frac{q^2}{2M^2}\frac{-\frac{q^2}{2}}{2E'E}\right\}$$

$$= \frac{8e^4}{q^4}2M^2E'E\left\{\cos^2\tfrac{\theta}{2} - \frac{q^2}{2M^2}\sin^2\tfrac{\theta}{2}\right\}$$



$$q = k - k'$$

$$M^2 = p'^2 = (p+q)^2$$

$$= p^2 + 2p\cdot q + q^2$$

$$= M^2 + 2M\nu + q^2$$

$$\nu = \frac{p\cdot q}{M} = E - E'$$

Cross section in lab frame:

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{E'}{ME}\right)^2 |M_{fi}|^2$$

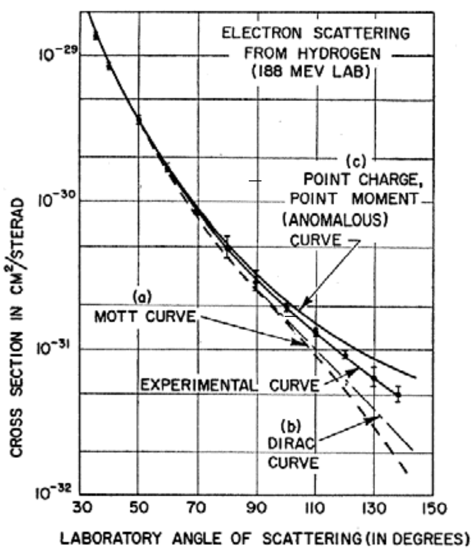
$$\downarrow e^2 = 4\pi\alpha$$

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2\sin^4\frac{\theta}{2}}\cdot\frac{E'}{E}\left\{\cos^2\frac{\theta}{2} - \frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\right\}$$

electric & magnetic interaction

3.5

e⁻-proton scattering



R.W. McAllister, R. Hofstadter, Phys.Rev. 102 , 851 (1956)

3.5

Form factors of the nucleon

Due to its finite size the proton needs to be described by 2 form factors, one for the charge distribution and one for the distribution of the magnetic moments.

$$\left(\frac{d\sigma}{d\Omega}\right)_{ep} = \left(\frac{d\sigma}{d\Omega}\right)_{Ruth} \frac{E'}{E}\left(\frac{G_E^2(q^2)+\tau G_M^2(q^2)}{1+\tau}\cos^2\frac{\theta}{2} + 2\tau G_M^2(q^2)\sin^2\frac{\theta}{2}\right)$$

$$\tau = -\frac{q^2}{4M^2} = \frac{Q^2}{4M^2} > 0$$

Generalized "Rosenbluth – formula" with so called Sachs - form factors

Static limit: $Q^2=0$

Electric form factor G_E $\qquad G_E(Q^2=0) = \frac{\text{charge}}{e}$

Magnetic form factor G_M $\qquad G_M(Q^2=0) = \frac{\mu_N}{\mu_K}$

Magnetic moments: $\vec{\mu} = g_N \mu_K \vec{s}$

Point like proton $\qquad \mu_K = \frac{e}{2M}$

Dirac particle: $\qquad \vec{\mu} = \frac{e}{M}\vec{S}$

$G_E^p(0) = 1$

$G_M^p(0) = 2.79$

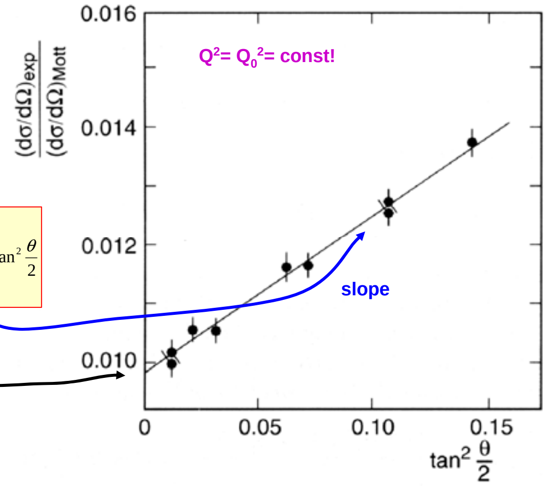
$G_E^n(0) = 0$

$G_M^n(0) = -1.91$

3.5

Measurement of $G_E(Q^2)$ and $G_M(Q^2)$

Rosenbluth - diagram



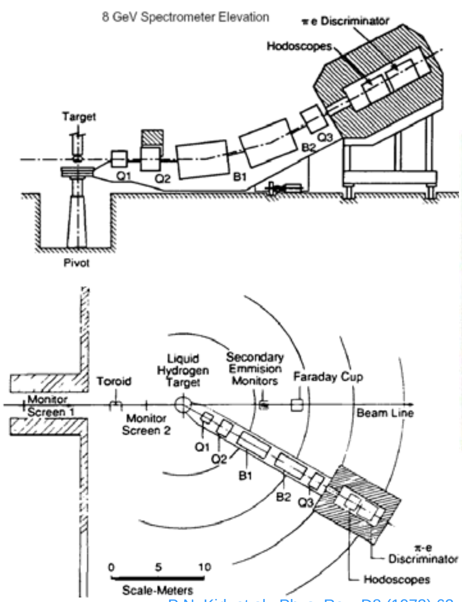
$$\frac{\left(\frac{d\sigma}{d\Omega}\right)_{ep}}{\left(\frac{d\sigma}{d\Omega}\right)_{Mott}} = \frac{G_E^2(Q^2)+\tau G_M^2(Q^2)}{1+\tau} + 2\tau G_M^2(Q^2)\tan^2\frac{\theta}{2}$$

slope

abzissa

3.5

SLAC Experiment (1967)



P.N. Kirk et al., Phys. Rev. D8 (1973) 63

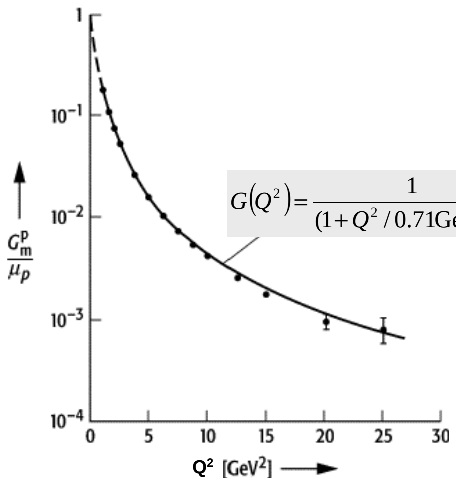
Spectrometer measurements
Determine differential cross section for different scattering angles at different incident energies but at fixed Q^2.

Experimental details:
Momentum resolution
Particle identification to suppress pion background

3.5

Magnetic form factor of the protons

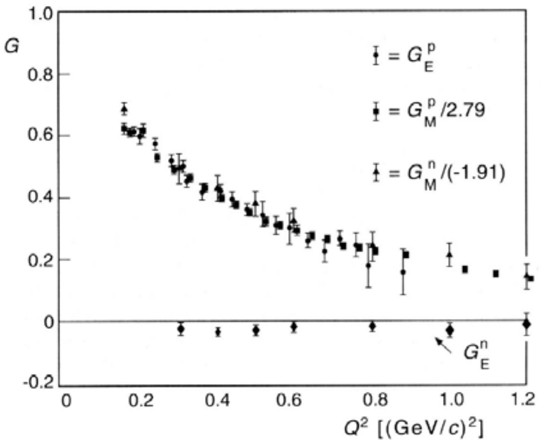
Good fit to the data with 'dipole form'



$$G(Q^2) = \frac{1}{(1 + Q^2 / 0.71\mathrm{GeV}^2)^2}$$

3.5

Form factors of nucleons



All formfactors except G_e^n show the same Q^2 dependence,

-> dipole-form.

Charge and magnetic moments have the same distribution in the proton.

Proton and neutron have the same size.

Determination of size by Fourier transformation:

$$\rho(r) = e^{-ar}, a = 4.27\ fm^{-1}$$

$$\langle r^2 \rangle = -6 \frac{dG}{dQ^2}\bigg|_{Q^2=0} = \frac{12}{a^2}$$

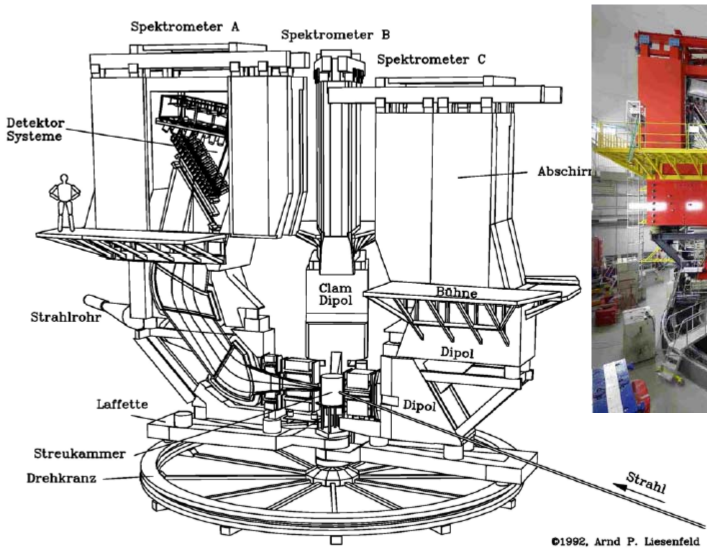
$$\sqrt{\langle r^2 \rangle} = 0.86\ fm$$

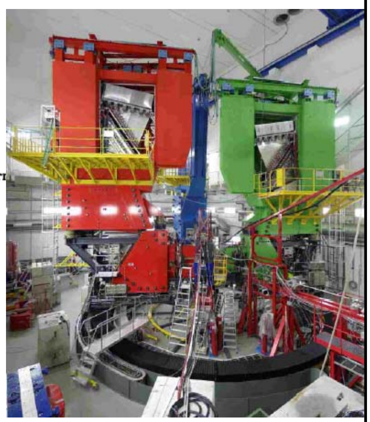
N.Herrmann, Uni Heidelberg 03/11/2011

3.5

Spectrometer design

MAMI A1 Kollaboration

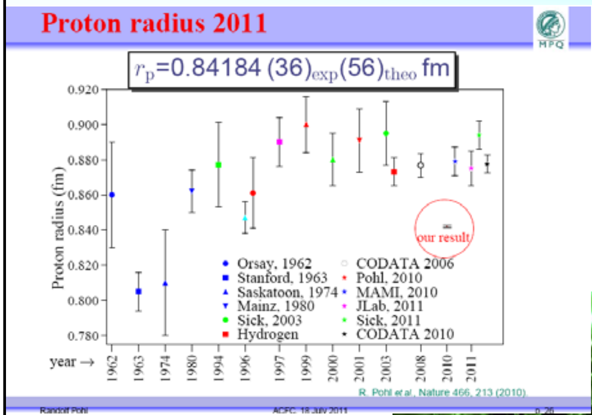




N.Herrmann, Uni Heidelberg 03/11/2011

3.5

How big is the proton?

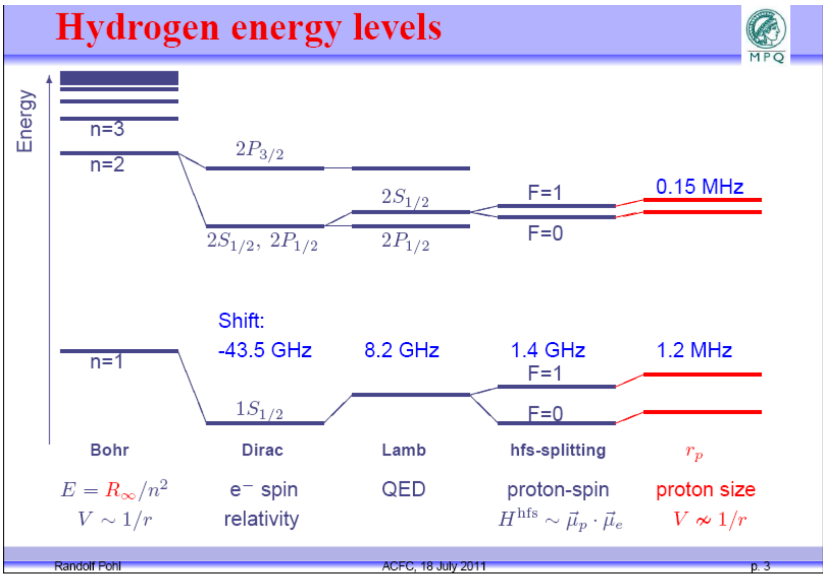


New spectroscopy measurements on muonic hydrogen in 2010.

Proton is smaller than inferred from electron scattering.

3.5

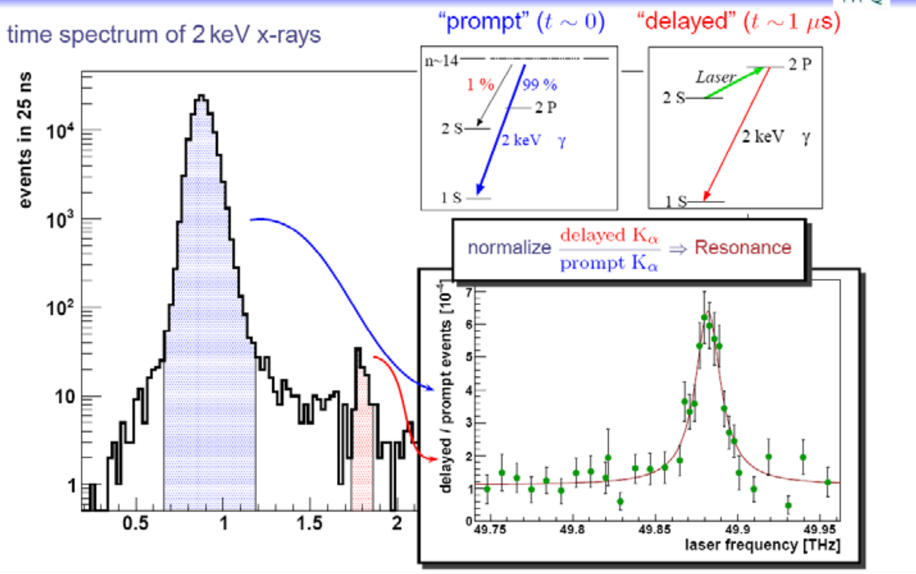
Lamb shift



3.5

Lamb shift experiment

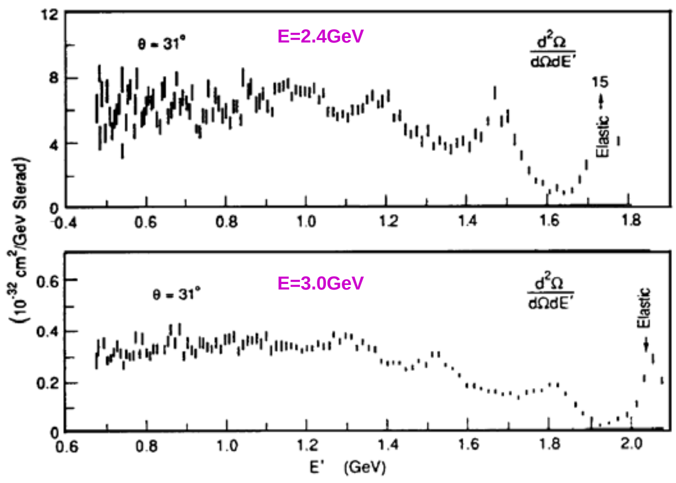
μp Lamb shift experiment: Principle



3.6

Inelastic ep – scattering

Electron energy spectrum at fixed polar lab angle:



R.Taylor, Nobel lecture (1990)